Vibration of conical shell frusta of variable thickness with fluid interaction

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ABSTRACT

Free vibration of layered conical shell frusta of thickness filled with fluid is investigated. The shell is made up of isotropic or specially orthotropic materials. Three types of thickness variations are considered, namely linear, exponential and sinusoidal along the radial direction of the conical shell structure. The equations of motion of the conical shell frusta is formulated using Love's first approximation theory along with the fluid interaction. Velocity potential and Bernoulli's equations have been applied for the expression of the pressure of the fluid. The fluid is assumed to be incompressible, inviscid and quiescent. The governing equations are modified by applying the separable form to the displacement functions and then it is obtained a system of coupled differential equations in terms of displacement functions. The displacement functions are approximated by cubic and quintics splines along with the boundary conditions to get generalized eigenvalue problem. The generalized eigenvalue problem is solved numerically for frequency parameters and then associated eigenvectors are calculated which are spline coefficients. The vibration of the shells with the effect of fluid is analysed for finding the frequency parameters against the cone angle, length ratio, relative layer thickness, number of layers, stacking sequence, boundary conditions, linear, exponential and sinusoidal thickness variations and then results are presented in terms of tables and graphs.
Keywords: conical shell; free vibration; love's first approximation theory; quiescent fluid; spline approximation; variable thickness

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1. INTRODUCTION

Love was first proposed Classical Shell Theory (CST) for bending analysis of shells which also include a linear analysis of thin shells (Love, 1994). Since the theory was based on Kirchhoff-Love assumption, then it is referred to as Love's first approximation theory. Since then, thin shell theories were developed through different assumptions and simplifications, such as Reissner, Naghdi, Sander and Flügge's theories (Leissa, 1973). By relaxing the normality condition, First Order Shear Deformation Theory (FSDT) were formulated (Reddy, 2004). By considering different theories of shells, research on vibration in shell structures having variable thickness has been conducted numerously by researchers. This includes the studies on vibration by applying the spline method on free vibration of layered circular cylindrical shells of variable thickness using extension of Love's first approximation theory (Viswanathan et al., 2010), free vibration of symmetric angle-ply laminated cylindrical shells of variable thickness using first order shear deformation theory (Viswanathan *et al*., 2011), vibration of antisymmetric angle-ply composite annular plates of variable thickness by implementing first order shear deformation theory (Nor Hafizah *et al*., 2018), a vibration of layered truncated conical shells of differently varying thickness using extension of Love's first approximation theory (Viswanathan and Navaneethakrishnan, 2005). Two layered truncated conical shells filled with quiescent fluid was studied (Nurul Izyan et al., 2017). Free vibration of symmetric angle-ply layered conical shell of variable thickness under shear deformation theory (Viswanathan, Javed and Aziz, 2013), free vibration of laminated conical shell of variable thickness that includes first order shear deformation and considers shells as antisymmetric angle-ply orientation (Viswanathan, Nor Hafizah and Aziz, 2018) and free vibration of antisymmetric angle-ply conical shells having non-uniform sinusoidal thickness variation under first order shear deformation (Javed *et al*., 2016).

By applying Ritz method, free vibration analysis of functionally graded spherical torus structure with uniform variable thickness along axial direction under first-order shear deformation theory (Gao *et al*., 2019). An analytical method was analysed for the free vibration of a fluid loaded (submerged) ring-stiffened conical shell with variable thickness in the low frequency range based on the Flugge theory and the governing equations of vibration of a ring-stiffened conical shell are developed in the form of a coupled set of the first order differential equations (Liu, Liu and Cheng, 2014). An analysis is presented for the free vibration of a truncated conical shell with variable thickness by use of the transfer matrix approach. The applicability of the classical thin shell theory is assumed and the governing equations of vibration of a conical shell are written as a coupled set of first order differential equations by using the transfer matrix of the shell (Irie, Yamada and Kaneko, 1982). Axisymmetric free vibrations of laminated conical shells with a linear thickness variation in the meridional direction using Rayleigh-Ritz procedure was adopted for the analysis and a classical thin shell theory was used (Sankaranarayanan, Chandrasekaran and Ramaiyan, 1987).

Spline method is one of approximate method in solving boundary value problem equations (Bickley, 1968). In general, spline functions are generally acknowledged by Schoenberg (1946). The spline method can be found in the literature (Schoenberg and

Whitney, 1953; Ahlberg, Nilson and Walsh, 1967; Greville, 1969). Studies on vibration of shell structures using the method of spline and the equations were formulated by Love's first approximation theory included axisymmetric free vibration of layered cylindrical shell filled with fluid (Nurul Izyan *et al*., 2021). Furthermore, vibration of symmetrically layered angle-ply (Nurul Izyan and Viswanathan, 2019) and cross-ply (Nurul Izyan *et al*., 2019) cylindrical shells filled with fluid the first order shear deformation theory was conducted. Using the same method, free vibration of cross-ply laminated plates based on higher-order shear deformation theory was presented (Javed *et al*., 2018).

The present study analyses the shell behavior of elastic truncated conical shells filled with quiescent fluid. The fluid is assumed to be inviscid and incompressible. The shell with variable thickness is considered. The thickness variations are assumed to be linear, exponential and sinusoidal along the radial di-rection. The layers are considered to be thin, elastic and specially orthotropic or isotropic and assumed to be bonded perfectly together and to move without interface slip. By applying Love's shell theory, the equations of motion of the truncated conical shell are coupled in the longitudinal, circumferential and transverse displacement components. The displacements components are assumed in a separable form and a system of coupled differential equations in displacement functions are obtained. Then, the displacement functions are approximated by splines which are cubic and quintic. Collocation with these splines yields a set of field equations together with the equations of boundary conditions. Hence, it reduces to a system of homogeneous simultaneous algebraic equations on the assumed spline coefficients which resulting to a generalized eigenvalue problem. The eigenvalue problem is solved using eigensolution technique to obtain as many eigenfrequencies as required, starting from the least. From the eigenvectors the spline coefficients are computed from which the mode shapes are constructed. Two layered shells with different types of material such as S-glass Epoxy (SGE), High Strength Graphite Expoxy (HSG), PRD-490 III epoxy (PRD) and Aluminium (Al) are considered. The parametric studies concerning the effects of relative layer thickness, semi cone angle and length ratio of the cone under different boundary conditions on the frequencies are presented. The variations in thickness also are considered.

2. THEORETİCAL FORMULATİON

2.1 Equations of shell

Consider a composite laminated truncated conical shell with an arbitrary number of layers, which are perfectly bonded together. Each layer is assumed to be homogeneous, linearly elastic and isotropic or specially orthotropic. r_a and r_b are the radii of the cone at its small and large ends, α is the semi-vertical angle and $\ell = b - a$ is the length of the cone along its generator. The wall thickness is denoted by *h*. The orthogonal coordinate system (x, θ, z) is fixed at its reference surface, which is taken to be at the middle surface and u, v and w are longitudinal, circumferential and transverse displacements. The radius of the cone at any point along its length is $r = x\sin\alpha$. The radius at the small end of the cone is $r_a = a\sin\alpha$ and the other end is $r_b = b\sin\alpha$.

Equations of motion of truncated conical shell included fluid is given as

$$\begin{aligned}
&\frac{\partial N_x}{\partial x}+\frac{1}{x}(N_x-N_\theta)+\frac{1}{x\sin\alpha}\frac{\partial N_{x\theta}}{\partial\theta}=\rho h\frac{\partial^2 u}{\partial t^2},\\
&\frac{\partial N_{x\theta}}{\partial x}+\frac{2}{x}N_{x\theta}+\frac{1}{x\sin\alpha}\frac{\partial N_\theta}{\partial\theta}+\frac{1}{x\tan\alpha}\frac{\partial M_{x\theta}}{\partial x}+\frac{2}{x^2\tan\alpha}M_{x\theta}\\
&\quad+\frac{1}{x^2\tan\alpha\sin\alpha}\frac{\partial M_\theta}{\partial\theta}=\rho h\frac{\partial^2 v}{\partial t^2},\\
&\frac{\partial^2 M_x}{\partial x^2}+\frac{2}{x}\frac{\partial M_x}{\partial x}+\frac{2}{x\sin\alpha}\frac{\partial^2 M_{x\theta}}{\partial x\partial\theta}+\frac{2}{x^2\sin\alpha}\frac{\partial M_{x\theta}}{\partial\theta}-\frac{1}{x}\frac{\partial M_\theta}{\partial x}\\
&\quad+\frac{1}{x^2\sin^2\alpha}\frac{\partial^2 M_\theta}{\partial\theta^2}-\frac{1}{x\tan\alpha}\frac{N_\theta}{r}=\rho h\left(\frac{\partial^2 w}{\partial t^2}-\frac{p}{\rho h}\right)
\end{aligned} \tag{1}$$

where N_x, N_θ and $N_{x\theta}$ are the force resultants, M_x, M_θ and $M_{x\theta}$ are the moments resultants defined as

$$\begin{aligned}
&(N_x, N_\theta, N_{x\theta})=\int_z(\sigma_x,\sigma_\theta,\sigma_{x\theta})\,dz,\\
&(M_x, M_\theta, M_{x\theta})=\int_z(\sigma_x,\sigma_\theta,\sigma_{x\theta})\,zdz,
\end{aligned} \tag{2}$$

The strain–displacement relations of truncated conical shell are as follows

$$\begin{aligned}
&\varepsilon_x=\frac{\partial u}{\partial x},\\
&\varepsilon_\theta=\frac{u}{x}+\frac{1}{x\sin\alpha}\frac{\partial v}{\partial\theta}+\frac{1}{x\tan\alpha}w,\\
&\gamma_{x\theta}=\frac{1}{x\sin\alpha}\frac{\partial u}{\partial\theta}+\frac{\partial v}{\partial x}-\frac{v}{x},\\
&\kappa_x=\frac{\partial\psi_x}{\partial x},\\
&\kappa_\theta=\frac{1}{x\sin\alpha}\frac{\partial\psi_\theta}{\partial\theta}+\frac{1}{x}\psi_x,\\
&\tau_{x\theta}=\frac{\partial\psi_\theta}{\partial x}-\frac{1}{x}\psi_\theta+\frac{1}{x\sin\alpha}\frac{\partial\psi_x}{\partial\theta}.
\end{aligned} \tag{3}$$

where

$$\psi_x = -\frac{\partial w}{\partial x},$$
$$\psi_\theta = \frac{1}{x\tan\alpha}v - \frac{1}{x\sin\alpha}\frac{\partial w}{\partial \theta}.$$

The stress-strain relations of *k*-th layer by neglecting the transverse normal stress and strain are defined as

$$\begin{pmatrix} \sigma_x^{(k)} \\ \sigma_\theta^{(k)} \\ \tau_{x\theta}^{(k)} \end{pmatrix} = \begin{pmatrix} Q_{11}^{(k)} & Q_{12}^{(k)} & 0 \\ Q_{12}^{(k)} & Q_{22}^{(k)} & 0 \\ 0 & 0 & Q_{66}^{(k)} \end{pmatrix} \begin{pmatrix} \varepsilon_x^{(k)} \\ \varepsilon_\theta^{(k)} \\ \gamma_{x\theta}^{(k)} \end{pmatrix}, \tag{4}$$

Applying Eq. (3) into Eq. (4) and then substituting into Eq. (2), the force and moment resultants can be obtained as

$$\begin{pmatrix} N_x \\ N_\theta \\ N_{x\theta} \\ M_x \\ M_\theta \\ M_{x\theta} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 \\ A_{12} & A_{22} & 0 & B_{12} & B_{22} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & B_{66} \\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 \\ B_{12} & B_{22} & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & B_{66} & 0 & 0 & D_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \gamma_{x\theta} \\ \kappa_x \\ \kappa_\theta \\ \tau_{x\theta} \end{pmatrix} \tag{5}$$

where A_{ij}, B_{ij} and D_{ij} are the extensional rigidities, the bending-stretching coupling rigidities and the bending rigidities defined by

$$A_{ij} = \sum_{k=1} Q_{ij}^k (z_k - z_{k-1}),\; B_{ij} = \frac{1}{2}\sum_{k=1} Q_{ij}^k (z_k^2 - z_{k-1}^2),$$
$$D_{ij} = \frac{1}{3}\sum_{k=1} Q_{ij}^k (z_k^3 - z_{k-1}^3).$$

with $i, j = 1, 2, 6.$ z_k is the distance from midsurface to the surface of *k*-th layer. For a thin shell, Q_{ij}^k is reduced stiffness defined as

$$Q_{11}^k = \frac{E_x^k}{1-\upsilon_{x\theta}^k\upsilon_{\theta x}^k},\; Q_{12}^k = \frac{\upsilon_{\theta x}^k E_x^k}{1-\upsilon_{x\theta}^k\upsilon_{\theta x}^k}, Q_{22}^k = \frac{E_\theta^k}{1-\upsilon_{x\theta}^k\upsilon_{\theta x}^k}, Q_{66}^k = G_{x\theta}^k. \tag{6}$$

The elastic coefficients A_{ij}, B_{ij} and D_{ij} corresponding to layers of uniform thickness wi th superscript '*c*' can easily be obtained as $A_{ij} = A_{ij}^c g(x),\; B_{ij} = B_{ij}^c g(x),\; D_{ij} = D_{ij}^c g(x),$ in which

$$A_{ij}^{c} = \sum_{k=1} \bar{Q}_{ij}^{k}(z_k - z_{k-1}),\ B_{ij}^{c} = \frac{1}{2}\sum_{k=1} \bar{Q}_{ij}^{k}(z_k^2 - z_{k-1}^2), D_{ij}^{c} = \frac{1}{3}\sum_{k=1} \bar{Q}_{ij}^{k}(z_k^3 - z_{k-1}^3) \text{ with } i, j = 1, 2, 6,$$

where z_k, z_{k-1} are boundaries of the *k*th layer.

The thickness of the *k*th layer is assumed in the form $h_k(x) = h_{0k}g(x),$ where h_{0k} is a constant thickness. If $g(x) = 1$, then the thickness becomes uniform. For variable thickness, g(x) is a function of x; $h_k(x) = h_0 g(x),$. Therefore, the thickness variation of each layer is assumed in the form

$$g(x) = 1 + C_\ell \frac{x}{\ell} + C_e \exp\left(\frac{x}{\ell}\right) + C_s \sin\left(\frac{\pi x}{\ell}\right). \tag{7}$$

If $(C_e = C_s = 0)$, then the thickness variation becomes linear. It can be written as $C_\iota = \frac{1}{\eta} - 1$, where η is the taper ratio $\frac{h_k(0)}{h_k(1)}$. If $(C_\iota = C_e = 0)$, then the excess thick ness varies exponentially. If $(C_\iota = C_e = 0)$, then the excess thickness varies sinusoi dally. The thickness of the layer at X = 0 is h_{0k} for the first and third cases, bu t the thickness is $h_{0k}(1 + C_e)$ for the second case.

The force resultants and moment resultants are expressed in terms of the longitudinal, circumferential and transverse displacements *u*, *v* and *w* of the reference surface. The displacement components u, v and w are assumed in the form of

$$\begin{aligned} u(x,\theta,t) &= U(x)\cos n\theta e^{i\omega t}, \\ v(x,\theta,t) &= V(x)\sin n\theta e^{i\omega t}, \\ w(x,\theta,t) &= W(x)\cos n\theta e^{i\omega t}, \end{aligned} \tag{8}$$

where x and θ are the longitudinal and rotational coordinates, ω is the angular frequency of vibration, t is the time and n is the circumferential node number.

2.2 Fluid Term

The fluid is assumed to be incompressible, inviscid and quiescent. The velocity potential satisfied the Laplace equation. Laplace equation is expressed in conical coordinates system (x, θ, α) (Paidoussis, 2004) (Nurul Izyan *et al.,* 2017)

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{2}{x}\frac{\partial \phi}{\partial x} + \frac{1}{x^2 \sin^2 \alpha}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{x^2 \tan \alpha}\frac{\partial \phi}{\partial \alpha} + \frac{1}{x^2}\frac{\partial^2 \phi}{\partial \alpha^2} = 0 \tag{9}$$

where ϕ is the velocity potential. Therefore

$$V_x = \frac{\partial \phi}{\partial x}, V_\theta = \frac{1}{x \sin \alpha} \frac{\partial \phi}{\partial \theta}, V_\alpha = \frac{1}{x} \frac{\partial \phi}{\partial \alpha}$$

where $V_{x,}$ V_θ and V_α are components of the fluid velocity.

Velocity potential is assumed as follows

$$\phi(x,\theta,\alpha,t) = R(\alpha)\varphi(x,\theta,t) \tag{10}$$

Using Bernoulli's equation, pressure exerted by the fluid on the shell wall is written as

$$p = -\rho_f \frac{\partial \phi}{\partial t} \tag{11}$$

where ρ_f is the density of the fluid. The impermeability condition is applied to ensure the contact between the shell and the fluid, as follows

$$\left. \frac{1}{x} \frac{\partial \phi}{\partial \alpha} \right|_{\alpha=a} = \left. \frac{\partial w}{\partial t} \right|_{\alpha=a} \tag{12}$$

Hence, Eq. (11) is represented as

$$p = -\rho_f kx \frac{\partial^2 W}{\partial t^2} \tag{13}$$

where

$$\begin{aligned} k = -(\alpha(175n^4\alpha^8 - 1470n^3\alpha^8 - 8400n^3\alpha^6 + 302400n^2\alpha^4 + 35280n^2\alpha^6 + 3509n^2\alpha^8 \\ - 2286n\alpha^8 - 7257600n\alpha^2 - 423360n\alpha^4 - 29760n\alpha^6 + 87091200)) / (n(175n^4\alpha^8 \\ - 2870n^3\alpha^8 - 8400n^3\alpha^6 + 302400n^2\alpha^4 + 85680n^2\alpha^6 - 30358n\alpha^8 - 7257600n\alpha^2 \\ - 1632960n\alpha^4 - 241440n\alpha^6 + 87091200 + 1693440\alpha^4 + 178560\alpha^6 + 14515200\alpha^2 \\ + 18288\alpha^8)) \end{aligned}$$

The following non-dimensional parameters are introduced

$$X = \frac{x-a}{\ell}, a \le x \le b \text{ and } X \in [0,1],$$
$$\lambda = \omega\ell\sqrt{\frac{R_0}{A_{11}}}; \text{a frequency parameter,}$$
$$\beta = \frac{a}{b}; \text{a length ratio,} \tag{14}$$
$$\gamma = \frac{h}{r_a}; \text{ ratio of thickness to radius,}$$
$$\gamma' = \frac{h_0}{a}; \text{ ratio of thickness to a length,}$$
$$\delta_k = \frac{h_k}{h}; \text{ a relative layer thickness of k-th layer.}$$

Here, h_k is the thickness of the *k*-th layer, *h* is total thickness of the shell, h_0 is the constant thickness and r_a is the radius of the small end of the cone.

By applying the non-dimensional parameters (Eq. 14), the equations of the motion of the coupled system in terms of U, V and W displacements are obtained in the form

$$\begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{15}$$

where L_{ij} are the differential operators (Nurul Izyan *et al.,* 2017).

Since third of Eq. (15) contains derivatives of third order in *U*, the form of Eq. (15) is not convenient to the solution procedure we propose to adopt. Hence, the equations are combined within themselves and a modified set of equations are derived. First of Eq. (15) is differentiated with respect to *X* and used to eliminate *U'''*(*X*) in third equation. The modified set of equations are given by

$$\begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L^*_{31} & L^*_{32} & L^*_{33} \end{pmatrix} \begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{16}$$

where L^*_{ij} are the updated differential operators (Nurul Izyan *et al.,* 2017).

3. SOLUTION PROCEDURE

3.1 Spline collocation method

Bickley (1968) presented a spline collocation approach over a two-point boundary value problem, bringing out its computational superiority to other schemes like Hermite interpolation. With his prediction that a lower order approximation may yields better accuracy than a global higher order approximation, he constructed his cubic spline over the mesh In this problem, the displacement functions $U(X), V(X)$ and $W(X)$ are approximated by cubic and quintic spline functions $U^*(X)$, $V^*(X)$ and $W^*(X)$ as follows

$$
\begin{aligned}
U^*(X) &= \sum_{i=0}^{2} a_i X^i + \sum_{j=0}^{N-1} b_j \left(X - X_j\right)^3 H\left(X - X_j\right), \\
V^*(X) &= \sum_{i=0}^{2} c_i X^i + \sum_{j=0}^{N-1} d_j \left(X - X_j\right)^3 H\left(X - X_j\right), \\
W^*(X) &= \sum_{i=0}^{4} e_i X^i + \sum_{j=0}^{N-1} f_j \left(X - X_j\right)^5 H\left(X - X_j\right).
\end{aligned}
\tag{17}
$$

Here, $H\left(X - X_j\right)$ is the Heaviside step function. *N* is the number of intervals in the range of $X \in [0,1]$ is divided. The points of division $X = X_s = \frac{s}{N}, (s = 0,1,2,...N)$ are chosen as the knots of the splines as well as the collocation points. Imposing the condition that the differential equations given by Eq. (16) are satisfied by these splines at the knots, a set of $3N+3$ homogeneous equations into $3N+11$ unknown spline coefficients $a_i, b_j, c_i, d_j, e_i, f_j (i = 0,1,2,3,4\,;\, j = 0,1,2,..., N-1)$ is obtained.

3.2 Boundary condition

The following boundary conditions are used to analyse the problem which are

i. Clamped-Clamped (C-C);

$$U = 0,\ V = 0,\ W = 0,\ \frac{dW}{dX} = 0 \text{ at } X = 0 \text{ and } X = 1.$$

ii. Clamped–Free(C–F);

$$U = 0,\ V = 0,\ W = 0,\ \frac{dW}{dX} = 0 \text{ at } X = 0,$$

$$N_x = 0,\ M_x = 0,\ V_x = Q_x + \frac{1}{x \sin\alpha} \frac{\partial M_{x\theta}}{\partial \theta} = 0,\ T_{x\theta} = N_{x\theta} + \frac{M_{x\theta}}{x \tan\alpha} = 0 \text{ at } X = 1.$$

By applying the boundary conditions, gives eight more equations on spline coefficients. Combining these eight equations with the earlier $3N+3$ homogeneous equations, we get $3N+11$ homogeneous equations in the same number unknowns. Thus, one obtains the generalized eigenvalue problem as follows

$$[M]\{q\}=\lambda^2[P]\{q\}, \tag{18}$$

where [*M*] and [*P*] are square matrices, {*q*} is a column matrix of the spline coefficients and λ is a frequency parameter.

4. RESULTS AND DISCUSSIONS

The variation of frequency parameter values with respect to relative layer thickness, cone angle and length ratio are analysed. The linear, exponential and sinusoidal variations in thickness of layers are taken into consideration. The variation of frequency parameter with respect to the relative layer thickness, along with the effect of including and neglecting the coupling between extensional and flexural vibration with variation in thickness of layers namely; linear (Fig. 1), exponential (Fig. 2) and sinusoidal (Fig. 3). Figs. 1-3 consist of plots of λ_m(m = 1,2, 3), where m is the meridional mode number, against δ. The continuous and dashed lines correspond respectively to the inclusion and neglect of the coupling effect between the longitudinal and flexural deflections, characterized by taking $B_{ij} \neq 0$ and $B_{ij} = 0$, respectively. Two layered shell at both ends of the cone are clamped were considered with material combination of HSG-SGE, HSG-PRD and St-SGE.

Thickness of the layers vary linearly $(C_\iota \neq 0, C_e = 0, C_s = 0)$ were considered as shown in Fig. 1. The taper ratio η, which is the ratio of the thickness of the shell at $x = a$ to its thickness at $x = b$, is 0.5. The semi cone angle α is 30^o. The length ratio β is 0.5; the shell considered to be of medium length, compared to short shells (large β) and long shells (small β). The thickness parameter γ is set equal to 0.05. Fig. 1(a) shows the frequency parameter with the shell made up of HSG-SGE materials. The inner and outer layers are made up of HSG and SGE materials, respectively. Thus, when $\delta = 0$, the inner layer disappears and the shell in homogeneous, made up of SGE; when $\delta = 1$, it is again homogeneous, made up of HSG. For $0 < \delta < 1$, both layers are present. Generally, the frequency parameter variation curves corresponding to the lowest meridional mode; $m = 1$ have the least undulations. As m grows, the undulations get more and more pronounced.

Exponential variations in thickness of layers $(C_\iota = 0, C_e \neq 0, C_s = 0)$ were analysed as depicted in Fig. 2. The parameter $\alpha = 30^0, \beta = 0.5, \gamma = 0.05$ and $C_e = 0.2$ were fixed. As for Fig. 3, the analysis on the sinusoidal variations in thickness of layers $(C_\iota = 0, C_e = 0, C_s \neq 0)$ was carried out. The parameter $\alpha = 30^0, \beta = 0.5, \gamma = 0.05$ and $C_s = 0.2$ were set. Both values of C_e and C_s has been taken to limit the range of values of these parameters so that the thickness does not vanish or become negative anywhere and the thin shell assumptions are valid.

In the case of HSG-PRD laminations, the layers are orthotropic and with the ratios E_x/ρ comparable, though not nearly same; and hence the frequencies are not much affected by relative thickness of layers (Fig. 1(b), Fig. 2(b) and Fig. 3(b)). For other material combinations; HSG-SGE (Fig. 1(a), Fig. 2(a) and Fig. 3(a)) and St-SGE (Fig. 1(c), Fig. 2(c) and Fig. 3(c)) pronounced variations in frequencies are observed. In the case of St-SGE shells, though steel and SGE possess very close E/ρ ratios ($2.696\text{x}10^7 m^2 s^{-2}$) and ($2.530\text{x}10^7 m^2 s^{-2}$), respectively yet the variation in frequency is considerable, perhaps due to the effect of material orthotropy in the case of SGE.

Generally, it is seen as a common feature that the neglect of this coupling results in increase of the values of the frequencies and the increment is negligibly small. The maximum change occurs for $0.2 < \delta < 0.6$. For nearly homogeneous conditions; $\delta \leq 0.1$ and $\delta \geq 0.9$, the differences are much less, vanishing for homogeneous materials ($\delta = 0$ or 1).

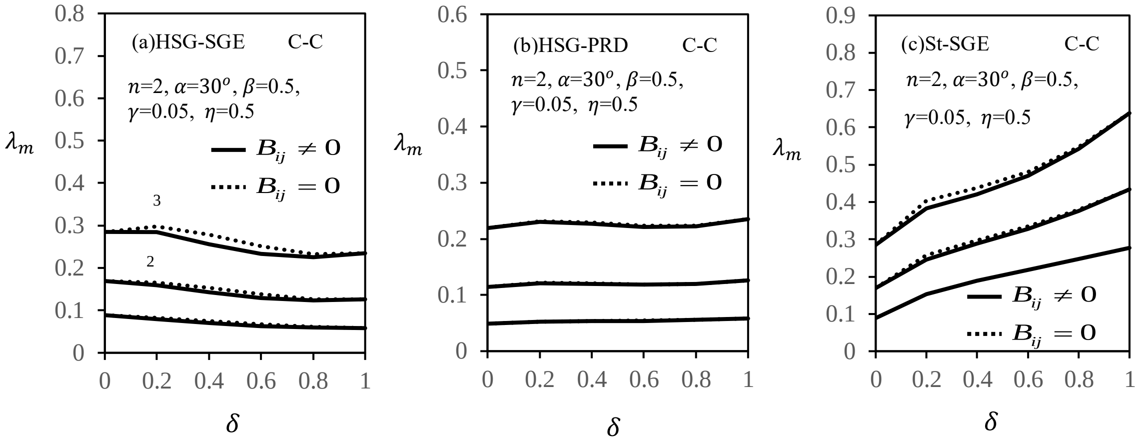

Fig. 1 Variation of frequency parameter with relative layer thickness and the effect of coupli[illegible] conical shells of linear variation in thickness of layers under C-C boundary conditions

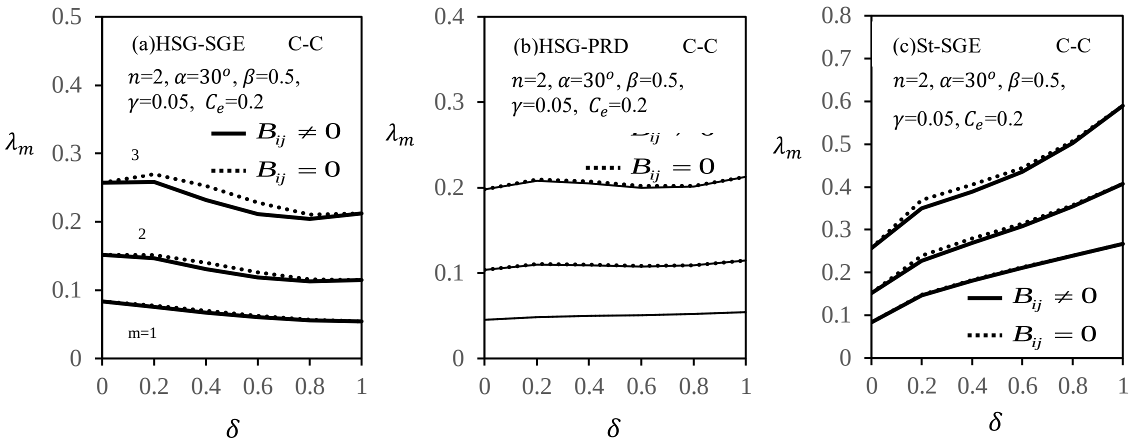

Fig. 2 Variation of frequency parameter with relative layer thickness and the effect of coupling: conical shells of exponential variation in thickness of layers under C-C boundary conditions

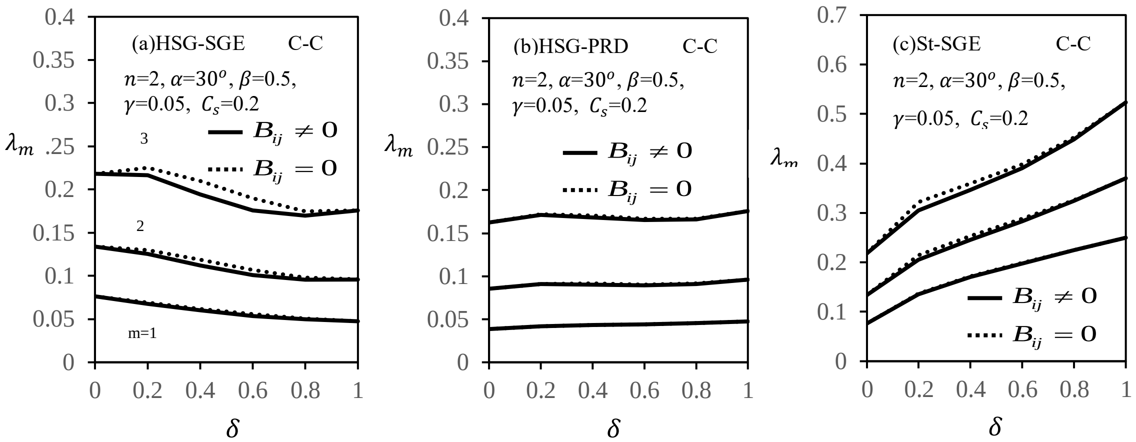

Fig. 3 Variation of frequency parameter with relative layer thickness and the effect of coupling: conical shells of sinusoidal variation in thickness of layers under C-C boundary conditions

Fig. 4 indicates the variation of the frequency parameter $\lambda_m (m=1,2,3)$ with respect to the semi cone angle under C-C boundary conditions. Linear ($\eta = 0.7$), exponential ($C_e = 0.15$), and sinusoidal ($C_s = 0.2$) variations are shown in Fig. 4(a), Fig. 4(b) and Fig. 4(c), respectively. The materials arranged in the order HSG-SGE with $n = 2,\ \beta = 0.5, \gamma' = 0.05$ and $\delta = 0.4$. When the cone angle α varies, $r_a = a \sin \alpha$ also varies, and hence $\gamma = h(a)/r_a$ cannot be held constant. Instead, another parameter $\gamma' = h(a)/a$ is considered. $\gamma = \gamma' \operatorname{cosec} \alpha$. The frequency parameter values are found to decrease with increasing cone angle. The decrease is rapid and almost constant up to $\alpha = 20^o$ for all cases considered. The same characteristic pattern of changes of frequencies with α is observed when the layers of the shells are varying in thickness (linear, exponential, and sinusoidal).

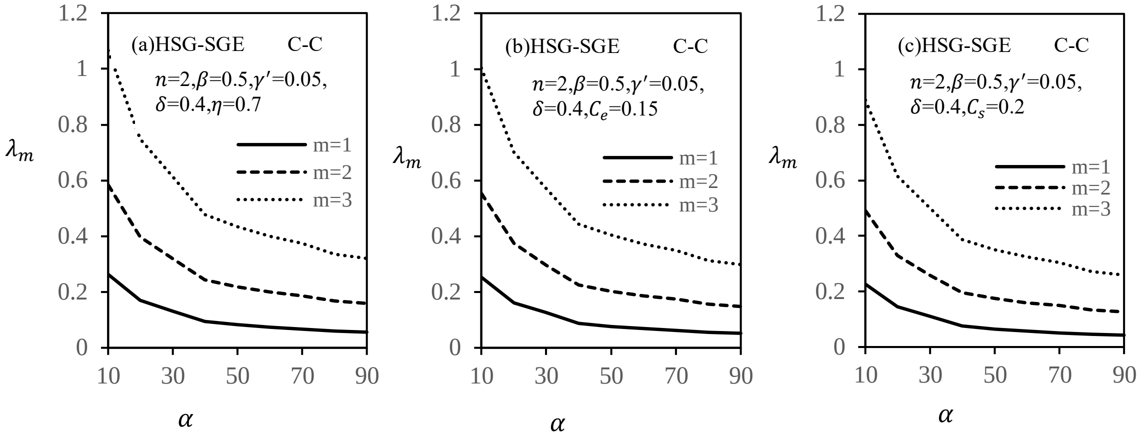

Fig. 4 Effect of cone angle on frequency parameter: (a) linear, (b) exponential, and (c) sinusoidal variations under C-C boundary conditions

The variation of angular frequencies, ω with respect to length of the cone under C-C boundary conditions with linear ($\eta = 0.7$), exponential ($C_e = 0.15$), and sinusoidal ($C_s = 0.2$) variations are shown in Fig. 5(a), Fig. 5(b) and Fig. 5(c), respectively. HSG-PRD material of two layered shell with $n = 2,\ \alpha = 30^0, \gamma = 0.05$ and $\delta = 0.4$ is set.

Since λ is a function of the length ℓ of the shell by definition, it may not be meaningful to study the variation of λ with β. Thus, the relation between the angular frequency, ω and β is studied. Since some length parameter must be given a specific value in such cases, $h(a)$=1cm is set for all cases considered. The frequencies increase with increase in β i.e., with decreasing cone-length. The increase is gradual and steady up to some value of β, and rapid afterwards. For very short shells (β>0.8), frequencies are very high.

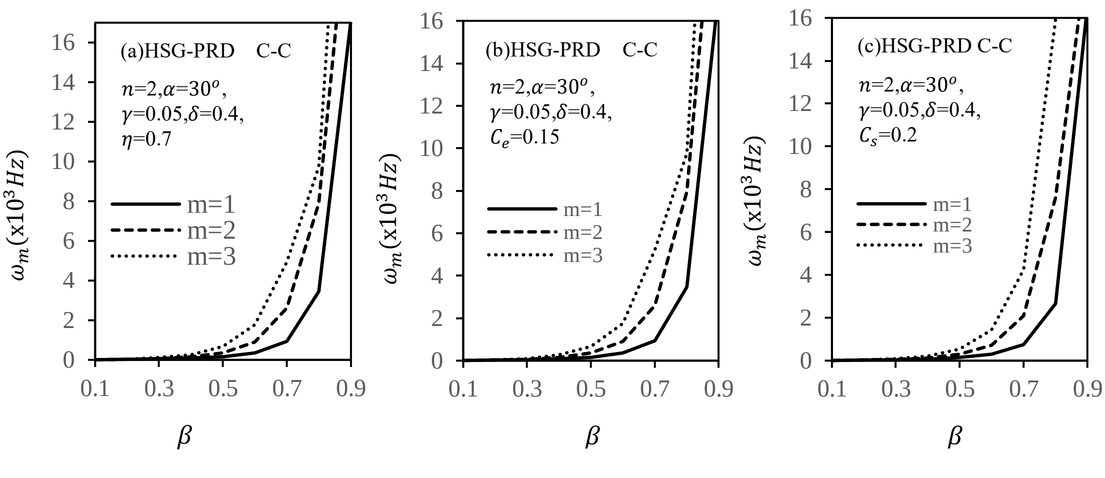

Fig. 5 Effect of length of cone on frequency parameter: (a) linear, (b) exponential, and (c) sinusoidal variations under C-C boundary conditions

The influence of the nature of variation of thickness of the layers of the shell on its frequency under different types of boundary conditions is shown in Fig. 6-8. Two layered shell with HSG-PRD material is used. $n = 2,\ \alpha = 30^0, \beta = 0.5,\ \gamma = 0.05$ and $\delta = 0.4$ is considered.

Fig. 6 corresponds to frequencies, λ_m(m = 1,2, 3) with respect to linear variation in thickness of layers in the range of $0.5 < \eta < 2.1$. When the taper parameter, $\eta = 1$, the thickness is constant. The thickness at the larger end of the cone is larger or smaller than the thickness at the smaller end, according as $\eta \lessgtr 1$. Result shows that λ_m(m = 1,2, 3) decreases with increase of η. This is aligned with the fact that the smaller the value of η, the larger is the thickness, resulting in higher stiffness. It can be seen that frequencies for all modes in C-C boundary conditions (Fig. 6(a)) is higher than C-F boundary conditions (Fig. 6(b)). The effect of λ_m is higher for higher mode. Also, the fundamental frequencies are least influenced in the case C-F boundary conditions. The curves are convex down for all the cases of Fig. 6.

The study on the effect of exponential variation in thickness of layers to frequencies, λ_m(m = 1,2, 3) under C-C and C-F boundary conditions as depicted in Fig. 7(a) and Fig. 7(b), respectively. When $C_e = 0$, thickness is uniform. The thickness at the wider end of the cone is higher or lower than the thickness at the other end according as $C_e \lessgtr 0$. This explains why the frequencies are highest and lowest at $C_e = \mp 0.25$, respectively. Frequencies increase as C_e increases. The effect of λ_m is higher for higher mode. The frequencies for all modes in C-C boundary conditions (Fig. 7(a)) is higher than C-F boundary conditions (Fig. 7(b)).

The effects of sinusoidal variation in thickness of layers on frequency parameters are analysed as shown in Fig. 8. C-C boundary conditions (Fig. 8(a)) while C-F boundary conditions (Fig. 8(b)) are considered. These effects are almost similar to exponential variation as discussed in Fig. (7).

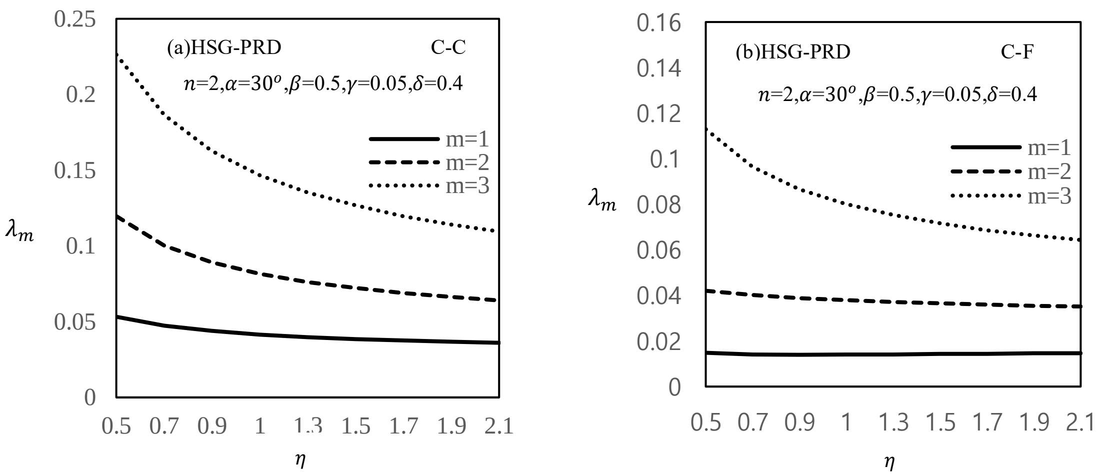

Fig. 6 Effect of taper parameter on frequency parameter

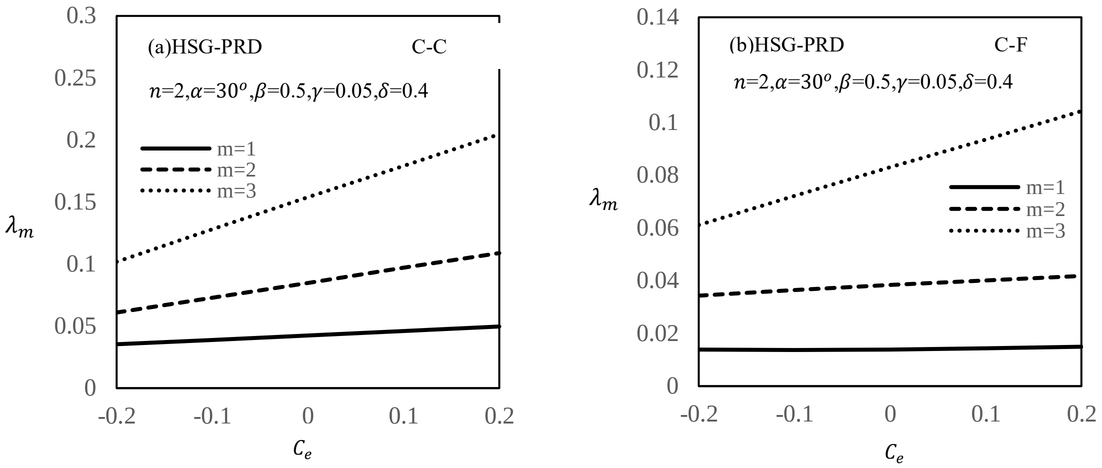

Fig. 7 Effect of coefficient of exponential variation of thickness of layers on fr equency parameter

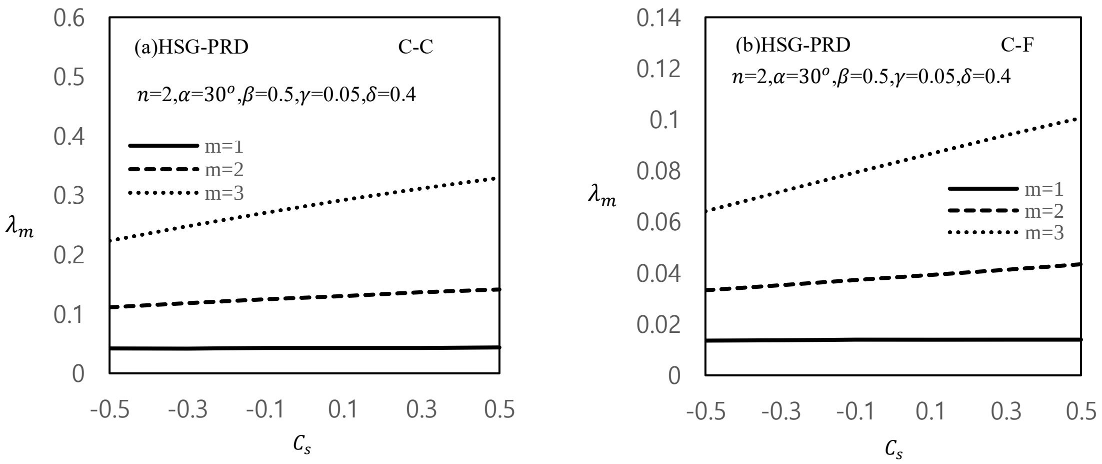

Fig. 8 Effect of coefficient of sinusoidal variation of thickness of layers on fre quency parameter

5. CONCLUSIONS

Vibration of two layered conical shell with the effect of fluid is investigated. The frequencies vary with relative layer thickness, the nature of variation of their thickness, stacking sequence, cone angle, length ratio and boundary conditions. The effect of neglecting the bending-stretching coupling is generally increase the frequencies with minimal increments. By a proper choice of δ, a desired frequency of vibration can be achieved.

Increasing cone angle led to decrease the frequencies in all cases. This effect is significant for lower modes and less significant for shorter shells. For any particular set of geometric parameters, the frequencies decrease as the length of the cone decrease. The effect is higher for higher modes.

The variation in thickness of layers (linear, exponential, sinusoidal) influence the natural frequencies of vibrations. For linear variation in thickness, the effect of taper ratio is high for both short shells and very small values of the taper ratio. In the case of exponential and sinusoidal variations in thickness of layers, the frequencies increase almost proportional to the corresponding coefficients of variation C_e and C_s. The rate of increase is higher for higher meridional modes.

By increasing the cone angle result in decreasing the frequencies while by reducing the length results in increasing the frequencies. These patterns are similar for all variations (linear, exponential, sinusoidal) in thickness of layers considered.

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