FREE VIBRATION OF LAYERED CYLINDRICAL SHELLS OF VARIABLE THICKNESS FILLED WITH FLUID

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Free vibrational behavior of layered cylindrical shell filled with quiescent fluid is investigated. Cylindrical shell with variable thickness, made up of isotropic or specially orthotropic materials is studied using spline approximation. The thickness variations are assumed to be linear, exponential and sinusoidal along the radial direction. The equations of motion are derived by extending Love's first approximation theory. Irrotational of an inviscid fluid are expressed as the wave equation. These two equations are coupled. The solutions of the displacement functions are assumed in a separable form to obtain a system of coupled differential equations in terms of the displacement functions. The displacement functions are approximated by Bickley-type splines. A generalized eigenvalue problem is obtained and solved numerically for the frequency parameter and an associated eigenvector of the spline coefficients. Two layered shells with different types of materials under Clamped-Clamped (C-C) boundary conditions are considered. The effect of relative layer thickness, length parameter, material properties, and coefficients of thickness variations on the frequency parameter is investigated.

Keywords: free vibration, variable thickness, love's first approximation theory, spline approximation

1. Introduction

Composite materials offer high strength, high stiffness and lightweight. In addition, composite materials also have the characteristics of corrosion resistance as well as better damping and shock absorbance. It can be designed to be far stronger than steel as it can be engineered to be strong in a specific direction. Composite materials can be found in automotive, construction and aircraft industries. Meanwhile, laminated composite consists of layers that is combined together to form a laminate. Each layer is called a ply or lamina. The lamina is the fundamental building block of laminated composite materials [1,2].

Classical Shell Theory (CST) was first proposed by Love for approximation of bending analysis of shells which also include a linear analysis of thin shells [3]. The theory was based on Kirchhoff-Love assumption, which is then commonly referred to as Love's first approximation theory. Later, many thin shell theories were developed through different assumptions and simplifications, such as Reissner, Naghdi, Sander and Flügge's theories [4]. First Order Shear Deformation Theory (FSDT) were then established by relaxing the normality condition [1]. Various researches in free vibration in shell structures has been conducted to determine the frequencies of the shells. Free vibration of the shell structures using different theory were studied. Research on free vibration of cylindrical shells filled with fluid under Love's theory was investigated using spline method [5-8]. Under FSDT, free vibration of cylindrical shells with antisymmetric angle-ply [9-10], cross-ply [11], symmetric angle-ply [12] were investigated. There are also studies on higher-order shear deformation theory [13]. Free vibration of layered cylindrical shells of variable thickness were analysed [14-16].

In this study, free vibration of cylindrical shell filled with fluid with variable thickness is investigated using spline method. Spline method is one of approximate method in solving boundary value problem [17]. The thickness variations are assumed to be linear, exponential and sinusoidal along the radial direction. The equations of motion are based on Love's first approximation theory. Irrotational of an inviscid fluid are expressed as the wave equation. These two equations are coupled. A generalized eigenvalue problem is solved numerically for the frequency parameter and an associated eigenvector of the spline coefficients. Two layered shells with different types of materials under Clamped-Clamped (C-C) boundary conditions are considered. The frequencies analysis on length parameter, material properties, and coefficients of thickness variations are studied.

2. Mathematical Formulation

2.1 Equations of shell

A thin layered circular cylindrical shell of having length ℓ, constant thickness h, radius r is considered. Each layer is assumed to be homogeneous, linearly elastic and isotropic or specially orthotropic. The x coordinate of the shell is taken along the longitudinal direction, θ and z coordinate are in the circumferential and radial direction respectively. Equations of motion for cylindrical shell coupled with fluid is written as

$$\begin{gathered}\frac{\partial N_x}{\partial x}+\frac{1}{r}\frac{\partial N_{\theta x}}{\partial \theta}=\rho h\frac{\partial^2 u}{\partial t^2},\quad \frac{\partial N_{x\theta}}{\partial x}+\frac{1}{r}\frac{\partial N_{\theta}}{\partial \theta}+\frac{1}{r}\frac{\partial M_{x\theta}}{\partial x}+\frac{1}{r^2}\frac{\partial M_{\theta}}{\partial \theta}=\rho h\frac{\partial^2 v}{\partial t^2},\\ \frac{\partial^2 M_x}{\partial x^2}+\frac{2}{r}\frac{\partial M_{\theta x}}{\partial x\partial \theta}+\frac{1}{r^2}\frac{\partial^2 M_{\theta}}{\partial \theta^2}-\frac{N_{\theta}}{r}=\rho h\left(\frac{\partial^2 w}{\partial t^2}-\frac{p}{\rho h}\right),\end{gathered} \tag{2.1}$$

where N_x, N_θ and $N_{x\theta}$ are the stress resultants, M_x, M_θ and $M_{x\theta}$ are the moments resultants and p is the pressure.

The fluid is assumed to be incompressible. Irrotational flow of an inviscid fluid undergoing small oscillations is expressed as wave equation. The equation of motion of the fluid can be written in the cylindrical coordinates system (*x, θ, r*) [18]

$$\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}+\frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2}+\frac{\partial^2 p}{\partial x^2}=\frac{\partial^2 p}{c^2\partial t^2} \tag{2.2}$$

where t is the time, p is the pressure and c is the sound of speed of the fluid. The x and θ -coordinates are the same as those of the shell, where the r-coordinate is taken from the x-axis of the shell.

The thickness of the kth layer is assumed in the form $h_k(x)=h_{0k}g(x)$, where h_{0k} is a constant thickness. In general, the thickness variation of each layer is assumed in the form $h_k(x)=h_0 g(x)$, and

$$g(x)=1+C_\ell\frac{x}{\ell}+C_e\exp\left(\frac{x}{\ell}\right)+C_s\sin\left(\frac{\pi x}{\ell}\right). \tag{2.3}$$

If $g(x)=1$, then the thickness becomes uniform. Therefore, A_{ij}, B_{ij} and D_{ij} corresponding to layers of uniform thickness with superscript 'c' can easily be obtained as $A_{ij}=A_{ij}^c g(x),\ B_{ij}=B_{ij}^c g(x),\ D_{ij}=D_{ij}^c g(x),$ in which $A_{ij}^c=\sum_{k=1}\bar{Q}_{ij}^k(z_k-z_{k-1}),\ B_{ij}^c=\frac{1}{2}\sum_{k=1}\bar{Q}_{ij}^k(z_k^2-z_{k-1}^2), D_{ij}^c=\frac{1}{3}\sum_{k=1}\bar{Q}_{ij}^k(z_k^3-z_{k-1}^3)$ with $i, j=1,2,6,$ where z_k, z_{k-1} are boundaries of the kth layer.

The displacement components u, v and w are assumed in the form of

$$u(x,t)=U(x)\cos n\theta e^{i\omega t},\ v(x,t)=V(x)\sin n\theta e^{i\omega t},\ w(x,t)=W(x)\cos n\theta e^{i\omega t}, \tag{2.4}$$

where x is the longitudinal, θ is the rotational, ω is the angular frequency of vibration, n is the circumferential node number and t is the time.

The following non-dimensional parameters are introduced

$$\begin{gathered}L=\frac{\ell}{r};\text{a length parameter, } X=\frac{x}{\ell};\text{ a distance coordinate, } \delta_k=\frac{h_k}{h};\text{ a relative layer thickness of k-th layer,}\\ H=\frac{h}{r};\text{the thickness parameter, } \lambda=\omega\ell\sqrt{\frac{R_0}{A_{11}}};\text{a frequency parameter, } R=\frac{r}{\ell};\text{a radius parameter.}\end{gathered} \tag{2.5}$$

Here r is the radius of the cylinder and h is the total thickness of the shell. Since only two layers is considered in this study, therefore, $\delta=\delta_1$ and $\delta_2=1-\delta_1$. The thickness of the kth layer of the shell is assumed in the form $h_k(X)=h_{0k}g(X)$. h_{0k} is a constant thickness. Therefore, $g(X)=1+C_\ell X+C_e\exp(X)+C_s\sin(\pi X)$. If $(C_e=C_s=0)$, then the thickness variation becomes linear. It can be written as $C_\iota=\frac{1}{\eta}-1$, where η is the taper ratio $\frac{h_k(0)}{h_k(1)}$. If $(C_\iota=C_e=0)$, then the excess thickness varies exponentially. If $(C_\iota=C_e=0)$, then the excess thickness varies sinusoidally. The thickness of the layer at X = 0 is h_{0k} for the first and third cases, but the thickness is $h_{0k}(1+C_e)$ for the second case.

2.2 Method of Solution

In obtaining equations of shell coupled with fluid, substituting Eq. (2.2) into stress and momentum resultants, then substituting into Eq. (2.1). Next, applying Eq. (2.4-2.5), the equations in the matrix form are obtained as follows

$$\begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix}\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{2.6}$$

where $L_{ij}\left(i = 1,2,3;\ j = 1,2,3\right)$ are the differential operators given as follows

$$L_{11} = \frac{d^2}{dX^2} + \frac{g'}{g}\frac{d}{dX} - S_{10}\frac{n^2}{R^2} + \lambda^2,\ L_{12} = \frac{n}{R}\left(S_2 + S_5\frac{1}{R}\right)\frac{g'}{g'} + \frac{n}{R}\left(S_2 + S_{10} + \frac{1}{R}\left(S_5 + S_{11}\right) + \right)\frac{d}{dX},$$

$$L_{13} = -S_4\frac{d^3}{dX^3} - S_4\frac{g'}{g}\frac{d^2}{dX^2} + \left(\frac{n^2}{R^2}\left(S_5 + 2S_{11}\right) + S_2\frac{1}{R}\right)\frac{d}{dX} + \frac{g'}{gR}\left(S_2 + S_5\frac{1}{R} + \right),$$

$$L_{21} = -\frac{n}{R}\left(S_2 + 2S_{10} + \frac{1}{R}\left(S_5 + 2S_{11}\right)\right)\frac{d}{dX} - \frac{n}{R}\left(S_{10} + S_{11}\frac{1}{R}\right)\frac{g'}{g},$$

$$L_{22} = \left(S_{10} + \frac{2S_{11}}{R} + \frac{S_{12}}{R^2}\right)\frac{d^2}{dX^2} + \left(S_{10} + \frac{2S_{11}}{R} + \frac{S_{12}}{R^2}\right)\frac{d}{dX} - \frac{n^2}{R^2}\left(S_3 + \frac{2S_6}{R} + \frac{S_9}{R^2}\right) + \lambda^2,$$

$$L_{23} = \frac{n}{R}\left(2S_{11} + S_5 + \frac{2S_{12}}{R} + \frac{S_8}{R}\right)\frac{d^2}{dX^2} + \frac{n}{R}\frac{g'}{g}\left(2S_{11} + \frac{2S_{12}}{R}\right)\frac{d}{dX} - \frac{n}{R^3}\left(\left(1 + n^2\right)S_6 + n^2\frac{S_9}{R}\right) - \frac{nS_3}{R^2},$$

$$L_{31} = S_4\frac{g'}{g}\frac{d^2}{dX^2} + \left(S_4\left(\frac{g'^2}{g^2} + S_{10}\frac{n^2}{R^2}\right)\frac{g''}{g'} - \frac{S_2}{R} - \frac{n^2}{R^2}\left(S_5 + 2S_{11}\right) - \lambda'^2 S_4\right)\frac{d}{dX} - 2S_{11}\frac{g'}{g}\frac{n^2}{R^2},$$

$$\begin{aligned} L_{32} = {} & \frac{n}{R}\left(2S_{11} + S_5 + \frac{1}{R}\left(S_8 + 2S_{12}\right)\right) - S_4\left(S_2 + S_{10} + \frac{1}{R}\left(S_5 + S_{11}\right)\right)\frac{d^2}{dX^2} + \frac{n}{R}\frac{g'}{g}\left(2\left(S_5 + S_{11} + \frac{S_8}{R} + \frac{S_{12}}{R}\right) - S_4\left(S_2 + \frac{S_5}{R}\right)\right)\frac{d}{dX} \\ & - \frac{n}{R}\left(\frac{n^2}{R^2}\left(S_6 + \frac{S_9}{R}\right) + \frac{S_3}{R} + \frac{S_6}{R^2} - S_5\frac{g''}{g} - \frac{S_8}{R}\frac{g''}{g} + S_4\left(S_2 + \frac{S_5}{R}\right)\left(\frac{g''}{g} - \frac{g'^2}{g^2}\right)\right), \end{aligned}$$

$$\begin{aligned} L_{33} = {} & \left(S_4^2 - S_7\right)\frac{d^4}{dX^4} + \left(S_4^2 - 2S_7\right)\frac{g'}{g}\frac{d^3}{dX^3} + \left(\frac{2S_5}{R} - S_7\frac{g''}{g} + \frac{n^2}{R^2}\left(4S_{12} + 2S_8\right) - S_4\left(\frac{n^2}{R^2}\left(S_5 + 2S_{11}\right) + \frac{S_2}{R} - S_4\left(\frac{g''}{g} - \frac{g'^2}{g^2}\right)\right)\right)\frac{d^2}{dX^2} \\ & + \frac{g'}{g}\left(\frac{2S_5}{R} + \frac{2n^2}{R^2}\left(S_8 + 2S_{12}\right) - S_4\left(S_5\frac{n^2}{R^2} + \frac{S_2}{R}\right)\right)\frac{d}{dX} - \begin{pmatrix} \frac{n^2}{R^2}\left(\frac{2S_6}{R} - S_8\frac{g''}{g}\right) + \frac{n^4}{R^4}S_9 + \frac{S_3}{R^2} - \frac{S_5}{R}\frac{g''}{g} \\ + S_4\left(S_5\frac{n^2}{R^2} + \frac{S_2}{R}\right)\left(\frac{g''}{g} - \frac{g'^2}{g^2}\right) \end{pmatrix} + \lambda^2\left(1 + \frac{\rho_f}{\rho_s h}\frac{J_n(R)}{J_n'(R)}\right). \end{aligned}$$

with

$$S_2 = \frac{A_{12}}{A_{11}},\ S_3 = \frac{A_{22}}{A_{11}},\ S_4 = \frac{B_{11}}{\ell A_{11}},\ S_5 = \frac{B_{12}}{\ell A_{11}},\ S_6 = \frac{B_{22}}{\ell A_{11}},\ S_7 = \frac{D_{11}}{\ell^2 A_{11}},\ S_8 = \frac{D_{12}}{\ell^2 A_{11}},$$

$$S_9 = \frac{D_{22}}{\ell^2 A_{11}},\ S_{10} = \frac{A_{66}}{A_{11}}, S_{11} = \frac{B_{66}}{\ell A_{11}},\ S_{12} = \frac{D_{66}}{\ell^2 A_{11}},\ \lambda^2 = \frac{R_0\omega^2}{A_{11}}, R_0 = \rho h.$$

2.3 Bickley-type Method

The spline approximation is a lower order approximation which yield a better accuracy than a global higher order approximation [17]. The displacement functions $U(X), V(X)$ and $W(X)$ are approximated by cubic and quintic spline functions $U^*(X), V^*(X)$ and $W^*(X)$, respectively as follows

$$\begin{aligned} U^*(X) &= \sum_{i=0}^{2} a_i X^i + \sum_{j=0}^{N-1} b_j \left(X - X_j\right)^3 H\left(X - X_j\right), V^*(X) = \sum_{i=0}^{2} c_i X^i + \sum_{j=0}^{N-1} d\left(X - X_j\right)^3 H\left(X - X_j\right), \\ W^*(X) &= \sum_{i=0}^{4} e_i X^i + \sum_{j=0}^{N-1} f_j \left(X - X_j\right)^5 H\left(X - X_j\right). \end{aligned} \tag{2.7}$$

Here, $H\left(X - X_j\right)$ is the Heaviside step function. N is the number of intervals in the range of $X \in [0,1]$ is divided. The points of division $X = X_s = \frac{s}{N}, (s = 0,1,2,...N)$ are chosen as the knots of the splines as well as the collocation points. Imposing the condition that the differential equations given by Eq. (2.7) are satisfied by these splines at the knots, a set of $(3N+3)$ homogeneous equations into $(3N+11)$ unknown spline coefficients $a_i, b_j, c_i, d_j, e_i, \quad f_j (i = 0,1,2,3,4; j = 0,1,2,..., N-1)$ are obtained. The Clamped-Clamped (C-C) boundary conditions are used to analyse the problem which is

$$U = 0, W = 0, \frac{dW}{dX} = 0 \text{ at } X = 0 \text{ and } X = 1.$$

Thus, Eq. (2.7) reduces to a system which is called as a generalized eigenvalue problem of the form

$$[M]\{q\} = \lambda^2 [P]\{q\}, \tag{2.8}$$

where $[M]$ and $[P]$ are matrices of order $(3N+7)\times(3N+7)$, $\{q\}$ is a matrix of order $(3N+7)\times 1$. λ is the eigenparameter and eigenvector is the spline coefficients.

3. Results and Discussions

Free vibration of layered circular cylindrical shells of variable thickness under clamped-clamped boundary conditions are analysed. The shells are considered as two-layered shells and two combinations of High Strength Graphite (HSG) and S-Glass Epoxy (SGE) materials are used. Convergence study has been carried out for the frequency parameters of two layered shells with fluid under C-C boundary conditions. It was found that the number of knots N could be taken as 14 since for the next value of N the percent change in the values of λ is very low, the maximum being 0.3%.

Fig. 1 depicts the variation of frequency parameter $\lambda_m (m = 1,2,3)$ with respect to the relative thickness δ under linear variation in thickness (η=0.75), exponential variation in thickness (C_e=0.2) and sinusoidal variation in thickness (C_s=0.25) as shown in Fig.1(a), Fig.1(b), Fig.1(c), respectively. The shells are clamped at both ends. The values of the circumferential number n, the ratio of the shell's constant thickness to radius H, and the ratio of the shell length to the radius L are fixed as 4, 0.02 and 1.5, respectively. The two layers of the shell are arranged in the order of HSG and SGE materials. When δ=0, the inner layer disappears, and the shell is homogeneous, which is made of SGE material. When δ =1 the outer layer disappears, again the shell is homogeneous, made of HSG material. It is clearly seen that as δ increase, λ_m decreases for $(m = 1,2)$ for all values of δ and $\lambda_m (m = 3)$ decrease for $0 < \delta < 0.8$ and a small increase for $0.8 < \delta < 1$.

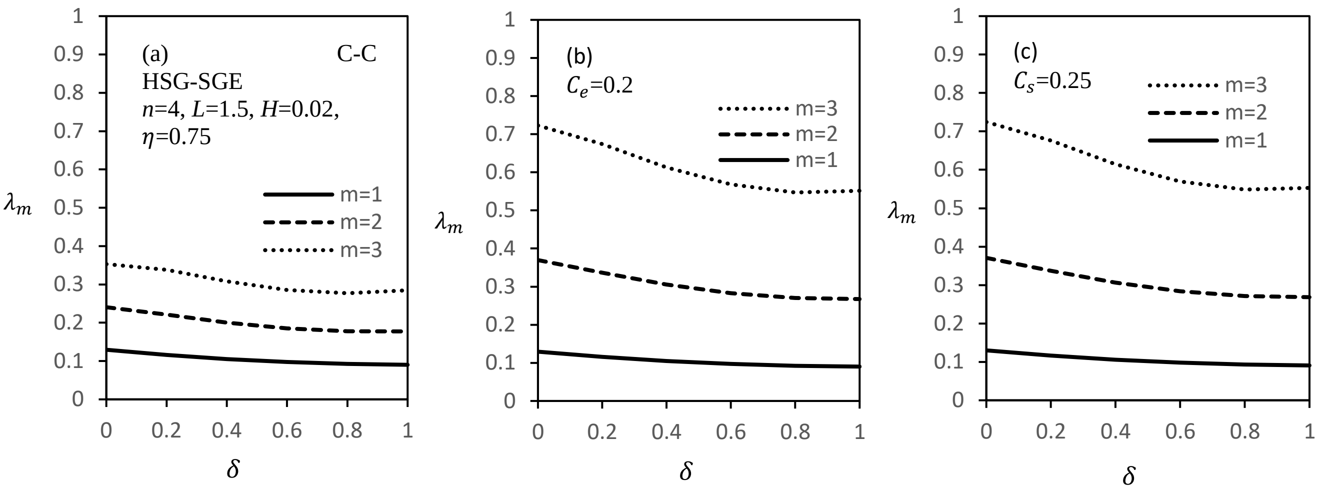

Figure 1: Variation of frequency parameter with relative layer thickness under C–C boundary conditions.

Fig. 2 shows the variation of the frequency parameter $\lambda_m (m = 1,2,3)$ with the thickness parameters η for two layered shells under C–C boundary conditions with H = 0.02, L = 1.5, $\delta = 0.4$ and n = 4 are fixed. The effect of linear (η=0.75), exponential (C_e=0.2) and sinusoidal (C_s=0.25) variation in thickness of layers on frequency parameters is shown in Fig. 2(a), Fig. 2(b) and Fig. 2 (c), respectively. The thickness is constant when the taper ratio η=1. It is seen that the values of $\lambda_m (m = 1,2,3)$ is almost constant for all thickness variation.

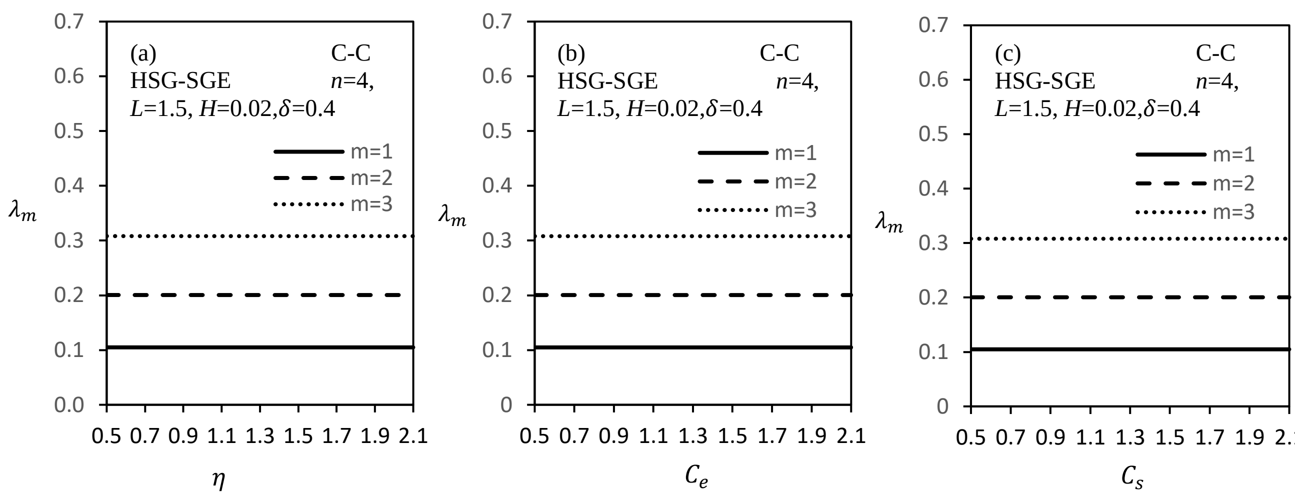

Figure 2: Effect of taper ratio, coefficient of exponential variation, and coefficient of sinusoidal variation on frequency parameter under C–C boundary conditions.

Fig. 3 illustrates the variation of angular frequencies ω on length parameter for two layered cylindrical shell with the materials arranged in the order *HSG-SGE* with $\delta = 0.4$, $H = 0.02$ and $n = 4$ under C-C boundary conditions. All the three types of variation in thickness of layers with η=0.75, C_e=0.2 and C_s=0.25 as shown in Fig. 3(a), Fig. 3(b) and Fig. 3(c), respectively. When studying the influence of the length of the cylinder for its vibrational behaviour, the angular frequency ω is considered instead of λ. In general, as L increases, ω decreases. In the range of $0.5 < L < 0.75$, the frequencies decrease fast. It can be observed that the angular frequencies decrease slowly in the range of $0.75 < L < 2$.

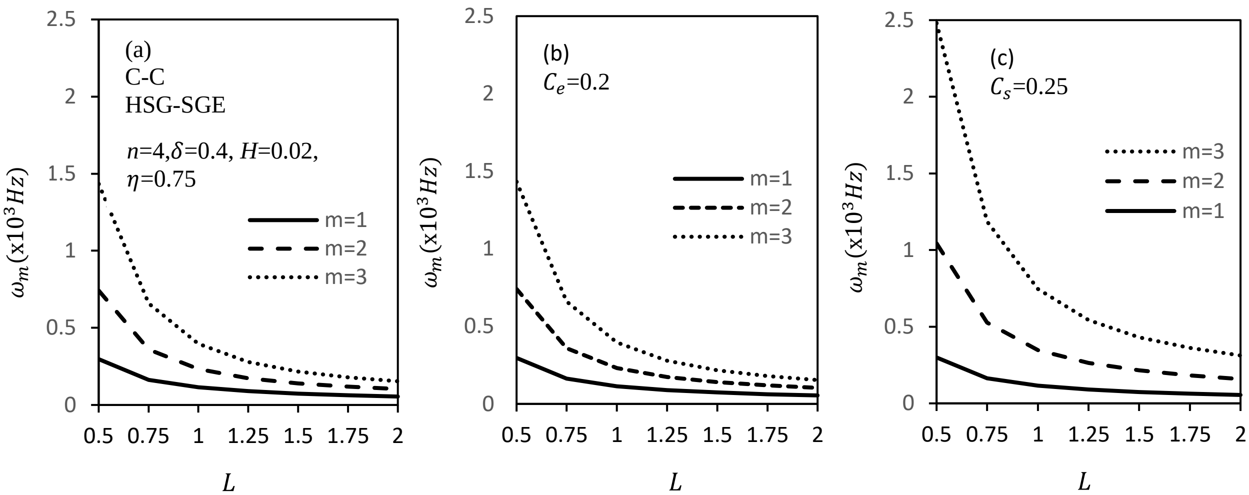

Figure 3: Effect of length of the shell on frequency parameter for different types of variation in thickness of layers under C–C boundary conditions.

4. Conclusion

Free vibration of two layered cylindrical shells of variable thickness is analysed using spline approximation with combination of HSG and SGE materials. The variation of frequencies with respect to the relative layer thickness, thickness coefficients, and length parameter are studied under C-C boundary conditions. The frequencies of the shells are significantly affected by material properties, length parameter, and different coefficient of thickness variations. It can be concluded that the frequency decreases as the length of the cylinder increases and it decreases fast in the range of $0.5 < L < 0.75$. Meanwhile, the frequency is almost constant for all coefficients of thickness variation.

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