A Quasi-Newton Like Method via Modified Rational Approximation Model for Solving System of Nonlinear Equation

Article *in* Journal of Advanced Research in Dynamical and Control Systems · June 2020

DOI: 10.5373/JARDCS/V12SP2/SP20201159

Kamilu Uba kamfa
Kano University of Science & Technology

Sulaiman Mohammed Ibrahim
Universiti Sultan Zainal Abidin | UniSZA

Mustafa Mamat
Universiti Sultan Zainal Abidin

A Quasi-Newton Like Method via Modified Rational Approximation Model for Solving System of Nonlinear Equation

Kamilu Kamfa[1], Muhammad Yusuf Waziri[2], Ibrahim Mohammed Sulaiman[3],
Mohd Asrul Hery Ibrahim[4], Mustafa Mamat[3,]*
[1]*Department of Mathematical Science, Kano University of Science and Technology, Wudil, Nigeria.*
[2]*Department of Mathematical Science, Faculty of Physical Sciences, Bayero, University Kano, Nigeria*
[3]*Faculty of Informatics and Computing Universiti Sultan Zainal Abidin, Terengganu, Malaysia.*
[4]*Faculty of Entrepreneurship and Business, Universiti Malaysia Kelantan, Kelantan, Malaysia*

Abstract: The major disadvantages of Newton method (NM) for nonlinear system of equations consist of finding and storing $n \times n$ Jacobin matrix which entail computation of the first derivatives in every iteration. In practice finding derivatives of some functions are often costly and sometime may not be available when the number of variables is large. In this paper, we proposed a new quasi newton like method via modified rational approximation model. Under certain condition, we show that this new algorithm has superlinear convergence rate. Numerical results have shown that the proposed method performed better than the classical methods for solving system of nonlinear equations.

Keywords: Rational Approximation Function, Newton, Local convergence

Introduction

Let consider the systems of nonlinear problems

$$F(x) = 0 \tag{1}$$

where $F: R^n \to R^n$ is continuously differentiable in an open neighborhood S of a solution $x_* \in S$. There are various iterative methods for solving (1). Newton method (NM) is the well-known method. This method generates a sequence of iterates $\{x_k\}$ using

$$x_{k+1} = x_k - J_k^{-1} F_k \tag{2}$$

where J_k is the Jacobian of F at x_k. The attractive property of this method is fast convergence rate and simple implementation. However, NM requires computation of the Jacobian matrix which consists the first-order derivatives of the systems. In practice, finding the derivatives of some functions are quite expensive, and sometimes may not even available.

Numerous researches have been carried out recently to improve the performance of this method using different approaches such as Jacobian approximation [1, 6], Diagonal update [8, 9] and improved Homotopy equation. Recently, Saheya et al. [5] proposed a new Newton-like method namely, improved Newton method (INM) based on Sui et al. work [10]. This method uses a rational approximation function with linear numerator and denominator to improve the Newtons iteration formula [5]. Despite this attractive feature, INM is computationally expensive, because it requires computing and storing of a full Jacobian matrix of $n \times n$ dimension in every iteration. This led to the idea of this paper. For recent study on numerical methods, please refer to [10-12].An application of the mathematical approach can be referred to [13].

In this paper, we present a new iterative algorithm via improved rational approximation model for solving (1). The anticipation has been to reduce the cost of computing and storing n $n \times n$ Jacobian matrix in every iteration which in turn will improve the overall performance of the INM. The rest of this paper is organized as follows. Section 2 entail the proposed method. Section 3 presents the convergence results. Numerical results are reported in section 4 and finally, conclusion comes in section 5.

Methodology

In this section, we present the new algorithm for solving (1), namely Derivative free Quasi Newton-like method via Modified Rational Approximation model (DFRM I). This new method is derivative free. An interesting feature of the

DOI: 10.5373/JARDCS/V12SP2/SP20201159
*Corresponding Author: Mustafa Mamat, E-mail: must@unisza.edu.my
Article History: Received: Mar 05, 2019, Accepted: May 26, 2020

method is that, when the modified rational model is zero, the propose method reduce to Broyden method (BM). The detailed description of the method is discussed below.

Saheya et al.; [5] improved the RALND function proposed by Sui et al. [10], so that it can be used to solve system of nonlinear equation. RALND function required finding function derivative in each step. In this paper, we design a new derivative free RALND function based on the structure of [5].
Let define the RALND function with the same horizon vector b_k for all nonlinear functions, $F_i(x), i = 1,2, \dots, n$ at x_k and approximate (1) by Rational function with linear denominator and numerator to get

$$F(x_k + s) \approx R(x_k + s) = F_k + \frac{B_k s}{1 + b_k^T s} \tag{3}$$

where $b_k, x_k \in R^n$, $s_k = x_k - x_{k-1}$ and x_k is the current point. Equation (3) is quite different from RALND function and approximate function in [5]. Because, this new approximation (3) use the same vector b_k for all function $F_i(x), i = 1,2, \dots, n$ at each iteration step x_k and it does not require any derivative of F_k at k iteration.

From (3) using linearization of Newton formula, we can derive a new iteration formula as follows,

$$(B_k + F_k b_k^T) s_k = F_k \tag{4}$$

Suppose the matrix $B_k + F_k b_k^T$ is nonsingular, then, using similar approach as in (2) we have

$$x_{k+1} = x_k - (B_k + F_k b_k^T)^{-1} F_k \tag{5}$$

where the matrix B_k is the Jacobian approximation, updated at each iteration. When b_k is zero, equation (5) reduced to the class of Quasi Newton method. Now, we define the vector $F_k b_k$. It is worthy to mention that at each step b_k is updated by requiring the following interpolation condition.

$$R(x_{k-1}) = F(x_{k-1}) \tag{6}$$

with this the search direction in (5) would depend on the Jacobian approximation B_k of the current point and the function values of the preceding point $F(x_{k-1})$ as well as the current point. Compare with BM where the search direction is determined by the B_k and x_k. Using the conic model [5], equation (3) and (6) become,

$$F_{k-1} = F_k - \frac{B_k s_{k-1}}{1 + b_k^T s_{k-1}} \tag{7}$$

Let the denominator of (7) be α_k i.e. $\alpha_k = 1 - b_k^T s_{k-1}$ and $y_{k-1} = F(x_k) - F(x_{k-1})$, then, (7) becomes

$$\alpha_k y_{k-1} = B_k s_{k-1}$$

thus

$$\alpha_k = \frac{y_{k-1}^T B_k s_{k-1}}{y_{k-1}^T y_{k-1}} \tag{8}$$

From (3), we can write the vector b_k as follows

$$b_k = \frac{(1-\alpha_k) c_k}{c_k^T s_{k-1}} \tag{9}$$

For any value of $c_k \in R^n$, such that $c_k^T s_{k-1} \neq 0$. Assuming $c_k = s_{k-1}$, then (9) become

$$b_k = \frac{(1-\alpha_k) s_{k-1}}{s_{k-1} s_{k-1}} = \frac{y_{k-1}(y_{k-1} - B_k s_{k-1}) s_{k-1}^T}{(y_{k-1}^T y_{k-1})(s_{k-1}^T s_{k-1})} \tag{10}$$

with (8) and (10), we have a new horizon vector $F_k b_k^T$ using similar linearization approach (2)

$$F_k b_k^T = \frac{(1-\alpha_k) s_{k-1}}{s_{k-1} s_{k-1}} = \frac{y_{k-1}(y_{k-1} - B_k s_{k-1})}{y_{k-1}^T y_{k-1}} \frac{F_k s_{k-1}^T}{s_{k-1}^T s_{k-1}} \tag{11}$$

The algorithm of the proposed scheme is presented as follows.

Algorithm 1: DFRM I algorithm

Initialization. Given x_0, let $B_0 = I$ for $k = 0$ choose ε_0.
Step 1: Compute $F(x_0)$ set $b_0 = 0$, $\varepsilon = \varepsilon_0$ and $k = 0$.
Step 2: If$\|s_k\| + \|F(x_k)\| \leq \varepsilon$, stop.
Step 3: Compute $F_{k+1}b_{k+1}^T$ and x_{k+1}, by

$$F_k b_k^T = \frac{y_{k-1}(y_{k-1} - B_k s_{k-1})F_k s_{k-1}^T}{y_{k-1}^T y_{k-1} s_{k-1}^T s_{k-1}}$$

$$x_{k+1} = x_k - A_k^{-1} F_k$$

updated B_k using Broyden formula

$$B_{k+1} = B_k + \frac{(y_k - B_k s_k)s_k^T}{(s_k^T s_k)}$$

note that $A_k = B_k + F_k b_k^T$

Step 4: Set $k = k + 1$ Go to Step 2

Unlike in INM, DFRM I method utilizes a derivative free rank one matrix to revise the Jacobian in every iteration and also it uses information of previous function to compute the next iteration point.

Convergence Analysis

In this section, we present the convergence result of DFRM I method. We make the following assumptions on nonlinear system F.

As usual we assume there exist $x_* \in R^n$ that solved (1), and J_k is continuous and nonsingular at x_* and there exist a constant $\mu > 0$ such that $\|J(x_*)\| \leq \mu$.

The following Lemmas are very useful in studying the convergence analysis of the proposed method.

Lemma 3.1. Let $F: R^n \to R^n$ be continuously differentiable on an open convex set $\Omega \in R^n, x \in \Omega$. If $J(x)$ is Lipchitz continuous with Lipchitz constant ρ, then for any $u, v \in \Omega$

$$\|F(u) - F(v) - J(x)(u - v)\| \leq \rho \max\{\|v - x\|, \|u - x\|\}\|v - u\|$$

Consequently, if $J(x)$ is nonsingular, then there exist ϵ and $\sigma > 0$, such that

$$\frac{1}{\sigma}\|v - u\| \leq \|F(v) - F(u)\| \leq \sigma\|v - u\|$$

For all $u, v \in \Omega$ for which $\max\{\|v - x\|, \|u - x\|\} \leq \epsilon$.

Lemma 3.2. Let $F: R^n \to R^n$, if $\{x_k\}$ converges p −superlinearly to$x_* \in R^n$,, then

$$\lim_{x\to\infty} \frac{\|x_{k+1} - x_k\|}{\|x_k - x_*\|} = 1$$

Lemma 3.3. Let $F: R^n \to R^n$ be continuously differentiable on an open convex set $\Omega \in R^n, x \in \Omega$. If $F'(x)$ is Lipchitz continuous with Lipchitz constant ρ, then for any $x + s \in \Omega$,

DOI: 10.5373/JARDCS/V12SP2/SP20201159
**Corresponding Author: Mustafa Mamat, E-mail: must@unisza.edu.my*
Article History: Received: Mar 05, 2019, Accepted: May 26, 2020

$$\|F(x+s)-F(x)-J(x)s\| \leq \frac{\gamma}{2}\|s\|^2$$

For the Proof of these lemmas please refer to [3].

Theorem 3.1. Let $F: R^n \to R^n$ satisfy the assumption of Lemma 3.1 on the set Ω. Let $\{B_k\}$ be a sequence of nonsingular matrices in R^n the space of real matrices of order n. Suppose for some x_0 the sequence $\{x_k\}$ generated by Algorithm 1 remain in Ω and $\lim_{k\to\infty} x_k = x_*$, where for each $k, x_k \neq x_*$. Then $\{x_k\}$ converges p −superlinearly to x and $F(x) = 0$ if and only if

$$\lim_{x\to\infty} \frac{\|(A_k - F'(x))s_k\|}{\|s_k\|} = 0 \tag{12}$$

Proof. Suppose equation (12) holds, then from step 3 of Algorithm 1, we have,

$$0 = (B_k + F_k b_k^T)s_k + F(x_k)$$
$$= (B_k + F_k b_k^T)s_k + F(x_k) - F'(x_*)s_k + F'(x_*)s_k$$

Thus

$$-F(x_{k+1}) = (B_k + F_k b_k^T - F'(x_*))s_k + (-F(x_{k+1})) + F(x_k) + F'(x_*)s_k \tag{13}$$

$$-F(x_{k+1}) = (B_k - F'(x_*))s_k + (F_k b_k^T)s_k + (-F(x_{k+1})) + F(x_k) + F'(x_*)s_k \tag{14}$$

Now, using vector norm properties and Lemma 3.1, we have

$$\frac{\|F(x_{k+1})\|}{\|s_k\|} \leq \frac{\|(B_k - F'(x_*))s_k\|}{\|s_k\|} + \frac{\|(F_k b_k^T)\|\|s_k\|}{\|s_k\|} + \frac{\|-F(x_{k+1}) + F(x_k) + F'(x_*)s_k\|}{\|s_k\|} \tag{15}$$

$$\frac{\|F(x_{k+1})\|}{\|s_k\|} \leq \frac{\|(B_k - F'(x_*))s_k\|}{\|s_k\|} + \|(F_k b_k^T)\| + \rho \max\{\|x_{k+1} - x_*\|, \|x_k - x_*\|\} \tag{16}$$

By induction,

$$\|(F_k b_k^T)\| = \|F_1\|\|b_1\| \leq \|F_1\| \left\| \frac{y_0^T(F_1 - F_0 - B_1 s_0)}{y_0^T y_0} \right\| \frac{1}{\|s_0\|}$$

$$\leq \|F_1\| \frac{\|y_0^T(F_1 - F_0 - B_1 s_0)\|}{\|y_0\|\|s_0\|} \leq \|F_1\| \frac{\gamma\|s_0\|}{2\|y_0\|}$$

$$\leq \frac{\gamma^2}{2\sigma}\|x_1 - x_*\|$$

In general, assuming the proof of the induction proceeds in a similar way, then

$$\|(F_k b_k^T)\| \leq \frac{\gamma^2}{2\sigma}\|x_1 - x_*\|$$

From (16) and $x_k \to x_*$, for all k

$$\lim_{k\to\infty} \frac{\|F(x_{k+1})\|}{\|s_k\|} \leq \lim_{k\to\infty} \frac{\|(B_k - F'(x_*))s_k\|}{\|s_k\|} + \lim_{k\to\infty} \frac{\gamma^2}{2\sigma}\|x_k - x_*\| \tag{17}$$

$$+\rho \lim_{k\to\infty} \max\{\|x_{k+1} - x_*\|, \|x_k - x_*\|\}$$

$$= 0.$$

$$F(x_*) = F(\lim_{k\to\infty} x_k)$$

$$= \lim_{k\to\infty} F(x_k)$$
$$= 0.$$

Since $F'(x_*)$ is invertible, by Lemma 3.1 there exist$\sigma > 0$, $k_0 \geq 0$, such that

$$\|F(x_{k+1})\| = \|F(x_{k+1}) - F(x_*)\| \geq \frac{1}{\sigma}\|x_{k+1} - x_*\| \quad (18)$$

Now, for all $k \geq k_0$ (17) and (18) become

$$0 = \lim_{k\to\infty} \frac{\|F(x_{k+1})\|}{\|s_k\|}$$

$$\geq \lim_{k\to\infty} \frac{1}{\sigma} \frac{\|x_{k+1} - x_*\|}{\|s_k\|}$$

$$\geq \lim_{k\to\infty} \frac{1}{\sigma} \frac{\|x_{k+1} - x_*\|}{\|x_k - x_*\| + \|x_{k+1} - x_*\|}$$

$$\frac{\frac{1}{\sigma}\theta_k}{1+\theta_k} where\ \theta_k = \frac{\|x_{k+1} - x_*\|}{\|x_k - x_*\|}$$

This show that $\lim_{k\to\infty} \theta_k = 0$, hence, $\{x_k\}$ converges $p -$ superlinearly to x_*.

Conversely, suppose that $\{x_k\}$ converges $p -$ superlinearly to $x_*$and $F(x_*) = 0$. By Lemma 3.1, there exist $\sigma > 0$ such that

$$\|F(x_{k+1})\| \leq \sigma\|x_{k+1} - x_*\|,$$

Now

$$0 = \lim_{k\to\infty} \frac{\|x_{k+1} - x_*\|}{\|x_k - x_*\|} \leq \lim_{k\to\infty} \frac{\|F(x_{k+1})\|}{\sigma\|x_k - x_*\|}$$

$$\lim_{k\to\infty} \frac{\|F(x_{k+1})\|\|s_k\|}{\sigma\|s_k\|\|x_k - x_*\|} \quad (19)$$

From equation (17), we have

$$\lim_{k\to\infty} \frac{\|(A_k - F'(x))s_k\|}{\|s_k\|} \leq \lim_{k\to\infty} \frac{\|F(x_{k+1})\|}{\|s_k\|} + \lim_{k\to\infty} \frac{\|(-F(x_{k+1})) + F(x_k) + F'(x_*)s_k\|}{\|s_k\|}$$

$$\leq 0 + \lim_{k\to\infty} \rho \max\{\|x_{k+1} - x_*\|, \|x_k - x_*\|\}$$

Since, $\{x_k\}$ converges to x_*, then, $\lim_{k\to\infty}\|x_k - x_*\| = 0$

$$\frac{\|(A_k - F'(x))s_k\|}{\|s_k\|} = 0,$$

which completes the prove. □

Numerical Results and Analysis

In this section, we present the numerical performance of DFRM I method when compared with Newton's method (NM), Improved Newton method (INM) and Broyden's method (BM). All the algorithms are coded in MATLAB R2013b and tested for some well-known benchmark problem with $\|s_k\| + \|F(x_k)\| \leq 10^{-6}$ as stopping condition. The performance profiles introduced by Dolan and More [4] is used to analyse the numerical results. This pro le gives a

DOI: 10.5373/JARDCS/V12SP2/SP20201159
*Corresponding Author: Mustafa Mamat, E-mail: must@unisza.edu.my
Article History: Received: Mar 05, 2019, Accepted: May 26, 2020

general information about the solver efficiency and probability of success in a concise way. In the following we give details of the test problems used in this work.

Problem 1. System of Nonlinear Equation [9]:

$$F_i(x) = (1 - x_i^2) + x_i(1 + x_i x_{n-2} x_{n-1} x_n) - 2$$

For $i = 1,2,\dots,n$

Problem 1. Spares 1 Function [9]:

$F_i(x) = x_i^2 - 1$
For $i = 1,2,\dots,n$

Problem 2. System of Nonlinear Equation [9]:

$$F_i(x) = x_i^2 - 1$$

For $i = 1,2,\dots,n$

Problem 3. Rosenbrock Function [8]:

$$F_i(x) = \begin{cases} 1 - x_i, & i = odd \\ 10x_{i+1} - x_i^2, & i = even \end{cases}$$

Problem 4. System of Nonlinear Equation [2]:

$$F_{3i-2}(x) = x_{3i} - 2x_{3i-1} - x_{3i}^2 - 1$$

$$F_{3i-1}(x) = x_{3i-2}x_{3i-1}x_{3i} - x_{3i-2}^2 + x_{3i-1}^2 - 2$$

$$F_{3i}(x) = e^{-x_{3i-2}} - e^{-x_{3i-2}}$$

For $\boldsymbol{i = 2,3,4 \dots, n}$.

From Table 1 the results show that of all the four methods tested, only DFRM I method is consistence, that is able to solve all the test functions. The performance of NM is quit impression but was not able to solved all the problem. In term of CPU time, it can be seen that, BM is better than both NM and INM in most cases. But, in term of Number of iterations, both methods outperform BM with good convergence rate. Also, Figure 1 and 2 presents the performance profile of all the methods. The DFRM I curve appears at the top, which signify it robustness and efficiency. It is not surprising because it required less computational cost when compared with NM and INM. is always superior to NM, INM and BM in term of Number of iteration and CPU time due to its low computational cost and memory demanding. NM and INM curve fall between DFRM I and BM.

Conclusion

In this paper, we present a new algorithm for solving small and medium system of nonlinear equations. It is a variant of secant method and uses modified rational approximation model to revise the Jacobian of the nonlinear system. Even though it is similar to INM, our algorithm is derivative free and can be used in place of INM particularly, when the derivative of the system is not available or too expensive to compute. Numerical results generated using the standard test problems, show in general, that our new algorithm is very effective and cheaper. In future we intend to extend the result to solved large scale system of nonlinear equation.

Acknowledgment

This work was funded by government of Malaysia using the grant number R/FRGS/A0100/01258A/003/2019/00670.

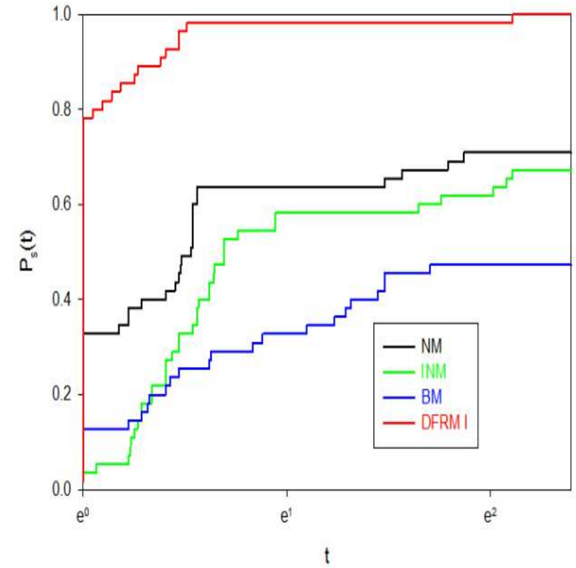

Figure 1: Performance Profile Based on Number of Iteration for Algorithm DFRM I

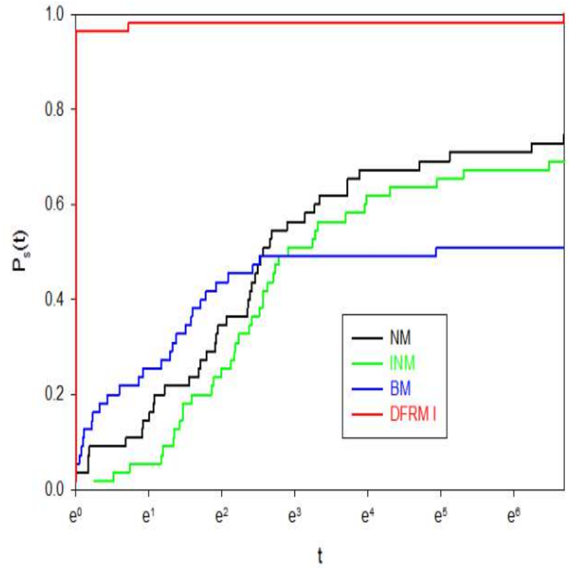

Figure 2: Performance Profile Based CPU Time for Algorithm DFRM I

Table 1: Numerical Results when using Algorithm 1

				NM			INM			BM			DRFM I	
Case	n	x_0	NI	CPU	$\|F(x_k)\|$	NI	CPU	$\|F(x_k)\|$	NI	CPU	$\|F(x_k)\|$	NI	CPU	$\|F(x_k)\|$
	3		10	0.005	3.27E-07	11	0.0035	5.06E-08	15	0.002	6.19E-09	7	0.0013	1.12E-11
	25		10	0.023	5.66E-07	12	0.0243	1.45E-10	15	0.0043	2.12E-07	6	0.0021	2.17E-08
1	120	0.1	11	0.1517	1.15E-10	12	0.1695	2.43E-10	15	0.0321	3.76E-07	6	0.0101	1.34E-08
	600		11	8.2444	3.10E-10	13	10.033	5.78E-10	15	1.1661	8.00E-07	6	0.2555	5.75E-08
	1200		11	67.8961	5.30E-10	15	93.414	1.64E-08	16	8.7939	1.42E-09	6	1.2696	8.29E-08
	2		11	0.0034	3.61E-09	17	0.0052	1.52E-10	18	0.0018	1.06E-07	6	0.0011	7.65E-12
	25		11	0.0166	1.28E-08	18	0.026	4.09E-09	21	0.0056	2.27E-07	6	0.0016	6.18E-12
2	120	0.01	11	0.107	2.84E-08	19	0.1973	1.09E-12	*	*	*	6	0.0097	2.15E-11
	600		11	7.1304	6.46E-08	19	12.606	5.92E-09	*	*	*	6	0.2447	1.36E-10
	1200		11	60.9246	9.27E-08	19	94.3548	4.42E-08	*	*	*	6	1.1451	1.99E-10
	2		*	*	*	*	*	*	24	0.0057	7.66E-07	14	0.0027	2.12E-09
	10		*	*	*	*	*	*	29	0.0087	2.71E-08	16	0.004	2.62E-08
3	100	0.6	*	*	*	*	*	*	33	0.0469	5.22E-08	21	0.0375	3.63E-08
	600		*	*	*	*	*	*	36	1.8312	2.68E-08	18	0.8341	2.16E-09
	1000		*	*	*	*	*	*	34	10.994	3.85E-07	18	3.7989	1.60E-09
	2		*	*	*	*	*	*	*	*	*	10	0.0016	5.36E-09
	25		*	*	*	*	*	*	*	*	*	10	0.0058	4.89E-09
4	120	0.1	*	*	*	*	*	*	*	*	*	11	0.0209	2.53E+00
	600		*	*	*	*	*	*	*	*	*	12	0.7269	9.45E-08
	1200		*	*	*	*	*	*	*	*	*	13	3.6832	8.67E-08

* Mean that Particular methods fail to converge

DOI: 10.5373/JARDCS/V12SP2/SP20201159
*Corresponding Author: Mustafa Mamat, E-mail: must@unisza.edu.my
Article History: Received: Mar 05, 2019, Accepted: May 26, 2020

References

[1] Waziri, M. Y., and Sabi'u, J. A derivative-free conjugate gradient method and its global convergence for solving symmetric nonlinear equations. *International Journal of Mathematics and Mathematical Sciences*, 2015.

[2] Dennis Jr, J. E., and Schnabel, R. B. *Numerical methods for unconstrained optimization and nonlinear equations*, Philadelphia: Siam, 1996.

[3] Dolan, E. D. and Moré, J. J., Benchmarking optimization software with performance profiles. *Math. Program.* vol. 91, no. 2, 2002, pp. 201–213.

[4] Saheya, B., Chen, G. Q., Sui, Y. K., and Wu, C. Y. A new Newton-like method for solving nonlinear equations. *SpringerPlus*, vol. 5, no. 1, 2016, art. Id. 1269.

[5] Drangoslav, H., and Natasa, K. Quasi-Newton's method with corrections. *Novi Sad Journal of Mathematics*, vol. 26, no. 1, 1996, pp. 115-127.

[6] Waziri, M. Y., June, L. W., and Hassan, M. A. Diagonal Broyden-like method for large-scale systems of nonlinear equations. *Malaysian Journal of Mathematical Sciences*, vol. 6, no. 1, pp. 59-73, 2012.

[7] Sui, Y., and Saheya, C. G. An improvement for the rational approximation RALND at accumulated two-point information. *Math Numer Sinica*, vol. 36, no. 1, 2014, pp. 51-64.

[8] Kamfa, K. U., Mamat, M., Abashar, A., Rivaie, M., Ghazali, P. L. B., and Salleh, Z. Another modified conjugate gradient coefficient with global convergence properties. *Applied Mathematical Sciences*, vol. 9, no. 37, 2015, pp. 1833-1844.

[9] Waziri, M. Y., Aisha, H. A., and Mamat, M. A structured Broyden's-Like method for solving systems of nonlinear equations. *Applied Mathematical Sciences*, vol. 8, no. 141, 2014, pp. 7039-7046.

[10] Sulaiman, I.M., Mamat, M., Puspa, L.G., Hazimi, M. F., Kamilu. K. Benny, P. An Improved Shamanskii Method for Solving Nonlinear Equation. *Jour of Adv Research in Dynamical & Control Systems*, vol. 12, no. 2, 2020, pp. 591-594.

[11] Umar, A.O., Sulaiman, I.M., Mamat, M., Hazimi, M.F., Altien, R., Dieby, S. New Hybrid Conjugate Gradient Method for Solving Fuzzy Nonlinear Equations. *Jour of Adv Research in Dynamical & Control Systems*, vol. 12, no. 2, 2020, pp. 585-590.

[12] Aini, N. Rivaie, M. Mamat, M. and Sulaiman, I. M. A Hybrid of Quasi-Newton Method with CG Method for Unconstrained Optimization. *J. Phys.: Conf. Ser*. vol. 1366, 2019. Art. Id. 012079.

[13] Ghazali,P.L., Mohd, I, Mamat, M., Ahmad, W.M.A.W. Mathematical Modelling in Family Takaful. Journal of Applied Sciences, 11(19), (2011), 3381-3388